Chapter 1:
Systems of Linear Equations

Sec. 1.1-1.3:
Solving Systems of Linear Equations Using Elementary Row Operations

Review: What is Meant by a Solution to an Equation?

Guess and check some potential solutions to: $2x = 10$

Guess	Statement	Solution?
$x = 3$	$2 \cdot 3 = 10?$ $6 = 10?$	No ✗
$x = 4$	$8 = 10?$	No ✗
$x = 5$	$10 = 10?$	Yes ✓
$x = 6$	$12 = 10?$	No ✗

Graph of all of the solutions:

Solutions look like: single numbers

Solution set = { 5 }

Review: What is Meant by a Solution to an Equation?

Guess and check some potential solutions to: $x + y = 5$

Guess	Statement	Solution?
(1,4)	$1 + 4 = 5?$ $5 = 5?$	Yes ✓
(2,3)	$5 = 5?$	Yes ✓
(3,4)	$7 = 5?$	No ✗
(5,0)	$5 = 5?$	Yes ✓

Graph of all of the solutions:

Solutions look like: ordered pairs

Solution set = $\{\ (5 - t,\ t)\ \mid\ t \in \mathbb{R}\ \}$

Review: What is Meant by a Solution to an Equation?

Guess and check some potential solutions to: $2x + y + 4z = 7$

Guess	Statement	Solution?
(1,1,1)	$2 \cdot 1 + 1 + 4 \cdot 1 = 7?$ $7 = 7?$	Yes ✓
(2,3,0)	$7 = 7?$	Yes ✓
$(2, -2, 4)$	$18 = 7?$	No ✗
$(2, 7, -1)$	$7 = 7?$	Yes ✓

Graph of all of the solutions:

Solutions look like: ordered triples

Solution set = $\{ \ (s, \ 7 - 2s - 4t, \ t) \ | \ s, t \in \mathbb{R} \ \}$

Review: What is Meant by a Solution to a SYSTEM OF EQUATIONS?

Finding a solution to a system of equations means finding a solution to ALL equations in the system.

Some solutions to $x + y = 5$: … (0,5) (1,4) (2,3) (3,2) (4,1) (5,0) (6,-1) …

Some solutions to $x - y = 3$: … (5,2) (4,1) (3,0) (2,-1) (1,-2) (0,-3) (-1,-4) …

What are the solutions to the system $\begin{matrix} x + y = 5 \\ x - y = 3 \end{matrix}$?

Solutions look like: ordered pairs

Graph: (Other Situations)

Solution set = { (4, 1) }

Review: What is Meant by a Solution to a SYSTEM OF EQUATIONS?

Finding a solution to a system of equations means finding a solution to ALL equations in the system.

Some solutions to $x + y + z = 5$: … (0,5,0) (2,2,1) (4,3,-2) …

Some solutions to $2x - y - z = 1$: … (2,2,1) (3,5,0) (0,-2,-1) …

Some solutions to $3x + 2y + 4z = 14$: … (4,1,0) (0,7,0) (2,2,1) …

What are the solutions to the system

$$\begin{aligned} x + y + z &= 5 \\ 2x - y - z &= 1 \\ 3x + 2y + 4z &= 14 \end{aligned}$$

?

Solutions look like: ordered triples

Graph: (Other Situations)

Solution set = { (2, 2, 1) }

Review: Quick Definitions

Definitions:

1) A system of equations is consistent if it has a solution (1 or more)

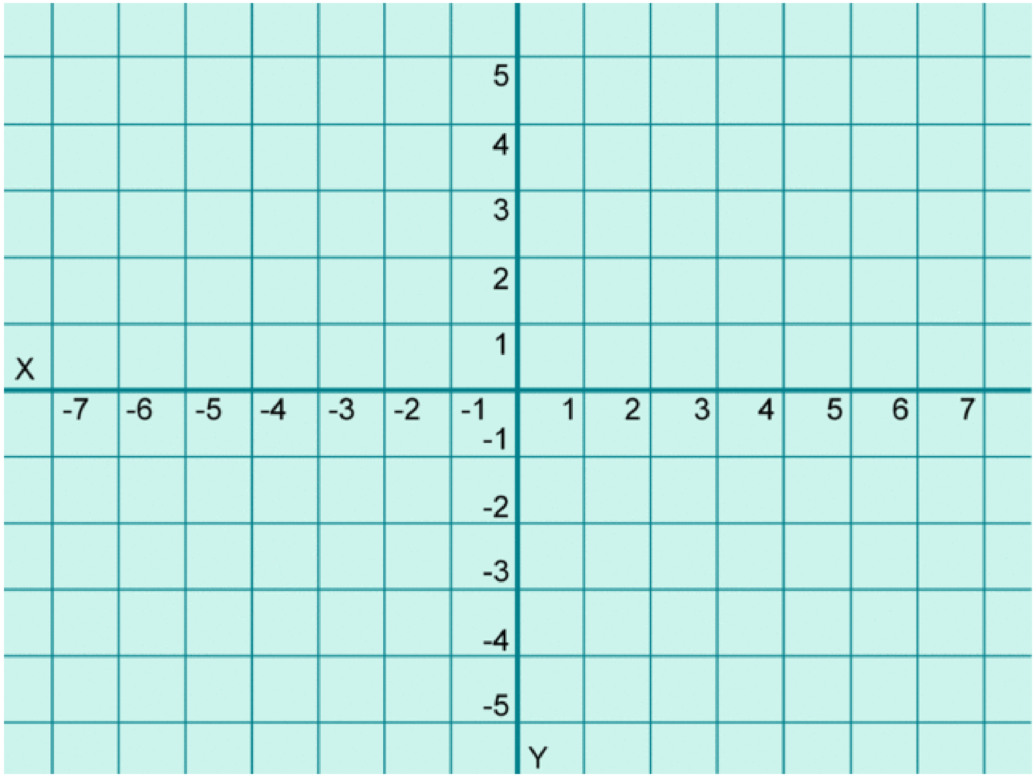
2) A system of equations is inconsistent if it has no solutions

Review: Solving Systems of Linear Equations (Beg. Alg. Methods)

In a beginning algebra course, you learn 3 ways to solve a system of linear equations…

1) Graphical Method

Solve $$x + y = 5$$
$$x - y = 3$$



Review: Solving Systems of Linear Equations (Beg. Alg. Methods)

In a beginning algebra course, you learn 3 ways to solve a system of linear equations…

2) Substitution Method

Solve $\quad (1)\ \ x + y = 5$
$\qquad\quad (2)\ \ x - y = 3$

- Turn 2 equations with 2 unknowns into 1 equation with 1 unknown
- Solve equation (2) for *x*

→ $\begin{array}{l} x - y = 3 \\ \quad {\color{red}+y \quad +y} \end{array}$ → $(3)\ \ x = y + 3$

- Plug equation (3) into equation 1

→ $(y + 3) + y = 5$

- Solve this 1 variable equation

→ $(y + 3) + y = 5$ → $2y + 3 = 5$ → $y = 1$

- Plug this value for *y* into any equation containing *x* to find the value of *x*

→ $x = y + 3$ → $x = 1 + 3 = 4$ → Solution = (4, 1)

Review: Solving Systems of Linear Equations (Beg. Alg. Methods)

In a beginning algebra course, you learn 3 ways to solve a system of linear equations…

3) The Addition Method (or the Elimination Method)

Solve $$\begin{array}{ll} (1) & x + 6y = 8 \\ (2) & 4x - 2y = 6 \end{array}$$

- Turn 2 equations with 2 unknowns into 1 equation with 1 unknown
- Multiply equation (2) by 3 then add the equations

$$\rightarrow \begin{array}{l} x + 6y = 8 \\ 4x - 2y = 6 \end{array} \quad \text{(mult. by 3)} \quad \rightarrow \quad \begin{array}{rl} & x + 6y = 8 \\ + & 12x - 6y = 18 \\ \hline & 13x \qquad = 26 \end{array}$$

- Solve this 1 variable equation

$$\rightarrow \quad 13x = 26 \quad \rightarrow \quad x = 2$$

- Plug this value for *x* into any equation containing *y* to find the value of *y*

$$\rightarrow \quad x + 6y = 8 \rightarrow \quad 2 + 6y = 8 \quad \rightarrow \quad y = 1 \quad \rightarrow \quad \text{Solution} = (2, 1)$$

Solving Systems of Linear Equations

Goal: Find all solutions to any system of linear equations (we will learn a procedure for this!)

Quick Definitions

<u>Def</u>:

3) An equation in variables $x_1, x_2, \ldots, x_n$ is a <u>linear equation</u> if it can be written in the form…

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers.

4) Two systems of equations are <u>equivalent</u> if they have exactly the same solutions set.

$$\begin{aligned} x_1 + x_2 &= 10 \\ x_1 - x_2 &= 4 \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} 2x_1 + x_2 &= 17 \\ -3x_1 + 2x_2 &= -15 \end{aligned}$$

Solution set for both = { (7,3) }

<u>Idea</u>: Keep replacing a system of linear equations with an equivalent one until eventually we end up with a system that is easy to solve.

Solving Systems of Linear Equations (Big Picture)

Solve…

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 6 \\ x_1 - 2x_2 + 5x_3 &= 4 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 2x_1 + x_2 - 2x_3 &= 6 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 5x_2 - 12x_3 &= -2 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 5x_2 - 12x_3 &= -2 \\ x_2 - 12x_3 &= -10 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ x_2 - 12x_3 &= -10 \\ 5x_2 - 12x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 \qquad - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ 5x_2 - 12x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 \qquad - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ 48x_3 &= 48 \end{aligned}$$

$$\begin{aligned} x_1 \qquad - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 \qquad &= 3 \\ x_2 - 12x_3 &= -10 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \\ x_3 &= 1 \end{aligned}$$

Solution Set = { (3,2,1) }

Solving Systems of Linear Equations (Big Picture)

In order to do less writing, we won't write the variables. Instead of writing a system of equations, we will represent the system with a matrix of coefficients and constants called an augmented matrix.

Ex 1: Write an augmented matrix corresponding to the system

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 6 \\ x_1 - 2x_2 + 5x_3 &= 4 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

Ex 2: Write a system of linear equations corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 3 & 5 & 8 & 1 \\ 1 & 0 & 4 & 7 & 3 \end{array}\right]$$

Solving Systems of Linear Equations (Big Picture)

Solve…

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 6 \\ x_1 - 2x_2 + 5x_3 &= 4 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 2x_1 + x_2 - 2x_3 &= 6 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 5x_2 - 12x_3 &= -2 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 5x_2 - 12x_3 &= -2 \\ x_2 - 12x_3 &= -10 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ x_2 - 12x_3 &= -10 \\ 5x_2 - 12x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ 5x_2 - 12x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ 48x_3 &= 48 \end{aligned}$$

$$\begin{aligned} x_1 - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 \\ x_2 - 12x_3 &= -10 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \\ x_3 &= 1 \end{aligned}$$

Solution Set = { (3,2,1) }

Solving Systems of Linear Equations (Big Picture)

Solve…

$$\begin{aligned} 2x_1 + \quad x_2 - 2x_3 &= \quad 6 \\ x_1 - 2x_2 + 5x_3 &= \quad 4 \\ 3x_1 - 5x_2 + 3x_3 &= \quad 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 6 \\ 1 & -2 & 5 & 4 \\ 3 & -5 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 2 & 1 & -2 & 6 \\ 3 & -5 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 0 & 5 & -12 & -2 \\ 3 & -5 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 0 & 5 & -12 & -2 \\ 0 & 1 & -12 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 0 & 1 & -12 & -10 \\ 0 & 5 & -12 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -19 & -16 \\ 0 & 1 & -12 & -10 \\ 0 & 5 & -12 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -19 & -16 \\ 0 & 1 & -12 & -10 \\ 0 & 0 & 48 & 48 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -19 & -16 \\ 0 & 1 & -12 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -12 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} x_1 \qquad\qquad &= 3 \\ x_2 \qquad &= 2 \\ x_3 &= 1 \end{aligned}$$

Solution Set = { (3,2,1) }

Solving Systems of Linear Equations (Big Picture)

So we have to answer 2 questions…

1) What are we allowed to do to a system of linear equations to get an equivalent system of linear equations?
   (What's the augmented matrix version of this?)

2) What kind of "easy" system of linear equations are we trying to get to?
   (What's the augmented matrix version of this?)

We'll answer the 2nd question first

Solving "Easy" Systems of Linear Equations

Ex 3: Solve

$$\begin{aligned} x_1 \quad\quad\quad &= 5 \\ x_2 \quad &= -3 \\ x_3 &= 2 \end{aligned}$$

Ex 4: Solve (back substitution)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 6 \\ x_2 - 4x_3 &= -6 \\ x_3 &= 1 \end{aligned}$$